RE-SCHEDULED PRMO-2019 SOLUTIONS

01. Consider the sequence of numbers $\left[n+\sqrt{2n}+\frac{1}{2}\right]$ for $n \geq 1$, where $[x]$ denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \ldots$ then find n_{12}.

Sol. Ans:91

$\left[n+\sqrt{2n}+\frac{1}{2}\right]=\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]$; If $n=1$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=2$;

If $n=2$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=4$; If $n=3$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=5$;

If $n=4$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=7$; If $n=5$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=8$;

If $n=6$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=9$; If $n=7$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=11$;

If $n=8$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=12$; If $n=9$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=13$;

If $n=10$, $\left[\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2\right]=14$

We observed that the sequence $\left(\sqrt{n}+\frac{1}{\sqrt{2}}\right)^2$ is : 2, 4, 5, 7, 8, 9, 11, 12, 13, 14,...

Missing numbers are: 3, 6, 10, 15, --------

We noticed that by adding +3, +4 , +5, ------ we are getting the sequence.

So, $n_1 < n_2 < n_3 < ...... = 3 < 6 < 10 < 15 < .....$

The required sequence is: 3, 6, 10, 15, 21, 28,36, 45, 55, 66, 78, 91

$n_{12} = 91$.

02. If $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

Sol. Ans: 93

Given $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$

$x - \sqrt{6} = \sqrt{2} + \sqrt{3}$

Square both sides of the above equation, we get,

$\left(x - \sqrt{6}\right)^2 = \left(5 + 2\sqrt{6}\right)$

$x^2 + 1 = 2\sqrt{6}(1 + x)$

Again square both sides of the above equation, we get,

$\left(x^2 + 1\right)^2 = \left[2\sqrt{6}(1 + x)\right]^2$

$x^4 + 2x^2 + 1 = 24\left(x^2 + 2x + 1\right)$

$x^4 - 22x^2 - 48x - 23 = 0$

Comparing this with $x^4 + ax^3 + bx^2 + cx + d = 0$

We get $a = 0, b = -22, c = -48, d = -23$.

$|a + b + c + d| = |0 - 22 - 48 - 23| = 93$.

03. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Sol. Ans: 25

N is difference of two squares

N is either odd or a multiple of 4.

Total positive integral numbers less than 101 which can be expressed as a difference of two squares = 75

(Odd numbers is 50 , multiples of 4 is 25.

Therefore, total number of numbers less than 101, that can be expressed as difference of two squares is 50+25=75.

Hence, the required number of numbers is equal to 100-75=25.

04. Let $a_1 = 24$ and form the sequence $a_n, n \geq 2$ by $a_n = 100a_{n-1} + 134$. the first few terms are

$24,\ 2534,\ 253534,\ 25353534, \ldots$

What is the least value of n for which a_n is divisible by 99?

Sol. Ans: 88

We observed that,

$a_1 = 24 \rightarrow$ No '53' between 2 & 4

$a_2 = 2\ \underline{53}\ 4 \rightarrow$ One 53 between 2 & 4

$a_3 = 2\ \underline{5353}\ 4 \rightarrow$ Two 53's between 2 & 4

So, in general

$a_n = 2\ \underbrace{53\,53\,53\ldots}_{n-1\ \text{times}}4$

Also we have $99 | a_n \Rightarrow 9 | a_n$ & $11 | a_n$

$9 | a_n \Rightarrow$ Sum of the digits should be divisible by 9.

$a_n = 2 + (5+3)(n-1) + 4 = 8n - 2$

So, $8n - 2$ divisible by 9 if $n \equiv 7 (\text{mod } 9)$

$11 | a_n \Rightarrow a_n = [2 + 3(n-1)] - [a + 5(n-1)] = -2n$

So, $2n$ is divisible by 11 if $n \equiv 0 (\text{mod } 11)$

This shows $n \equiv 7(\text{mod } 9)$ & $n \equiv 0(\text{mod } 11)$,

We get $n \equiv 88(\text{mod } 99)$ (By Chinse Remainder Theorem)

Least possible $n = 88$.

Alternate solution:

$a_1 = 24$

$a_n = 100a_{n-1} + 134$

$\Rightarrow a_n = a_{n-1} + 35(\text{mod } 99)$

$\Rightarrow a_n = a_{n-2} + 35 \times 2(\text{mod } 99)$

$\Rightarrow a_n = 24 + 35 \times (n-1)(\text{mod}\, 99)$

So we have to find minimum value of n

Such that 99 divides $24 + 35n - 35 = 35n - 11$

$99|35n - 11 \Rightarrow 11|35n - 11 \Rightarrow 11|\text{n}$

So, n must be a multiple of 11

$n = 11k$

$99/35(11k) - 11$

$\Rightarrow 9/35k - 1$

$35k - 1 \equiv -k - 1(\text{mod}\, 9)$

$\Rightarrow k \equiv -1(\text{mod}\, 9) \Rightarrow k = 8(\text{mod}\, 9)$

$\therefore$ The smallest +ve value of K that satisfy the above equation is 8.

$K = 8.$

$\Rightarrow n = 11.8 = 88.$

05. Let N be the smallest positive integer such that $N + 2N + 3N + \ldots + 9N$ is a number all whose digits are equal. What is the sum of the digits of N?

Sol. Ans: 37

$N + 2N + 3N + \ldots. + 9N = N(1 + 2 + 3 + \ldots. + 9) = 45 \times N$

Notice that all digits should be 5.

$N = \frac{555}{45}$ not an integer

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$N = \frac{555555555}{45}$ $= 12345679$ is the smallest

Sum of digits of $N = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 9 = 37.$

Alternate solution:

$$45N = \frac{a}{9}\left(10^k - 1\right)$$

$$9 \times 9 \times 5N = a\left(10^k - 1\right)$$

So, $d \mid a\left(10^n - 1\right)$

$\Rightarrow s \mid a$

Since $0 \le a \le 9$

$a = 5$

$$9 \times 9N = \left(10^k - 1\right) \Rightarrow 9N = \left(\underbrace{111...1}_{k\text{ times}}\right)$$

$\underbrace{1+1+1+1+...+1}_{k\text{ times}} = k$ must be divisible by 9

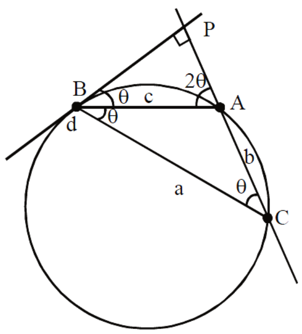
$$N = \frac{111\,111\,111}{9} = 12345679$$

$\therefore$ Sum of the digits of $N = 1+2+3+4+5+6+7+9 = 37$.

06. Let ABC be a triangle such that $AB = AC$. Suppose the tangent to the circumcircle of ΔABC at B is perpendicular to AC. Find $\angle ABC$ measured in degrees.

Sol. Let tangent at B and AC intersect of P.

As $AB = AC \Rightarrow \angle ABC = \angle ACB$



Let $\angle ABC = \angle ACB = \theta$

PB is a tangent by alternate segment theorem, we have $\angle PBA = \angle ACB = \theta$

Also, ext. $\underline{|BAC} = 2\theta$

In ΔPBA,

$90 + \theta + 2\theta = 180$

$\theta = 30^0$.

07. Let $s(n)$ denote the sum of the digits of a positive integer n in base 10. If $s(m) = 20$ and $s(33m) = 120$. What is the value of $s(3m)$?

Sol. Ans: 60

$s(ab) \le s(a)s(b)$ is true for any a, b

$s(33m) \le s(11)s(3m) \Rightarrow 120 \le 25(3m) \Rightarrow 60 \le s(3m)$

Also, $s(3m) \le s(3)s(m) \Rightarrow s(3m) \le 60$

So, $60 \le s(3m) \le 60$

So, $s(3m) = 60$

08. Let $F_k(a,b) = (a+b)^k - a^k - b^k$ and let $S = \{1,2,3,4,5,6,7,8,9,10\}$. For how many ordered pairs,(a, b) with $a, b \in S$ and $a \le b$ is $\dfrac{F_5(a,b)}{F_3(a,b)}$ an integer?

Sol. Ans: 22

$$\frac{F_5(a,b)}{F_3(a,b)} = \frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$$

$$= \frac{a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 - a^5 - b^5}{a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3}$$

$$= \frac{5ab(a^3 + 2a^2b + 2ab^2 + b^3)}{3ab(a+b)}$$

$$= \frac{5}{3}\frac{(a+b)^3 - ab(a+b)}{a+b}$$

$$= \frac{5}{3}\left[(a+b)^2 - ab\right] = \frac{5}{3}(a^2 + ab + b^2)$$

We have $3 \mid a^2 + ab + b^2$

Case:1

If $a \equiv 0 \pmod{3}$ and If $b \equiv 0 \pmod{3}$

Then $(a,b) = (3,3),(6,6),(9,9)(3,6),(3,9),(6,9)$

Case:2

If $a \equiv 1 \pmod{3}$ and If $b \equiv 1 \pmod{3}$

Then $(a,b) = (1,1),(1,4),(1,7),(1,10),(4,4),(4,7),(4,10),(7,7),(7,10),(10,10)$

Case:3

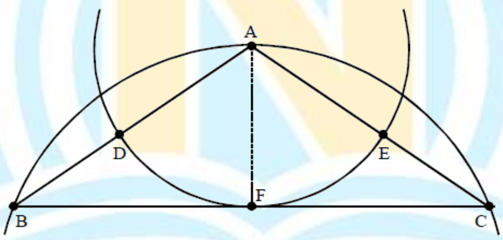
If $a \equiv 2 \pmod{3}$ and If $b \equiv 2 \pmod{3}$

Then $(a,b) = (2,2),(2,5),(2,8),(5,5),(5,8),(8,8)$

Total possibilities $= 6+10+6 = 22$

09. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α^0 and the smaller one β^0 then what is the value of $\alpha - \beta$?

Sol. Ans: 90



Let ABC be isosceles with $AB = AC$ and

D & E are mid poins of AB & AC respectively.

Let F be the midpoint of BC. which is the circumcenter of ΔABC.

In $ABC \Rightarrow AF \perp BC$ (The circle through D, E, F is tangent at F)

Notice that A is the centre of the circle through D, E, F $(AF \perp BC)$

We have $AF = AD = AE\,(Radius)$

$$\Rightarrow AB = 2AD = 2AF \Rightarrow \frac{AF}{AB} = \frac{1}{2} = Sin30^0$$

So, in right angled ΔABF, $\underline{|B} = 30^0$ and $\underline{|BAF} = 60^0$, So $\underline{|BAC} = 120^0$

$\alpha - \beta = 120 - 30 = 90^0$.

10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two-digit number. When I returned, it was y minutes past 6'O clock, and I noticed that

(i) I walked exactly for x minutes and

(ii) y was 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

Sol. Ans: 42

Let $x = \overline{ab}$

$x = 10a + b \Rightarrow y = 10b + a$

Time after 5 O'clock $= 5 \times 60 + 10a + b = 300 + 10a + b$

Time after 6 O'clock $= 6 \times 60 + 10b + a = 360 + 10b + a$

Given, $(360 + 10b + a) - (300 + 10a + b) = 10a + b.$

$\Rightarrow 60 + 9b - 9a = 10a + b \Rightarrow 19a - 8b = 60$

$b = \dfrac{19a - 60}{8}$

b is an integer when $a = 4$. So, $x = 42$.

11. Find the largest value of a^b such that the positive integers, $a, b > 1$ satisfy $a^b b^a + a^b + b^a = 5329$.

Sol. Ans: 81.

Given, $a^b b^a + a^b + b^a = 5329$

$\Rightarrow a^b b^a + a^b + b^a + 1 = 5330 \Rightarrow (a^b + 1)(b^a + 1) = 5330 = 2 \times 5 \times 13 \times 41$

Case:1 65×82

Case: 2 82×65

Case 1: $a^b + 1 = 65$, $b^a + 1 = 82$

$a^b = 64$, $b^a = 81$

$a = 4, b = 3$

$\therefore a^b = 4^3 = 64$

Case: 2 $a^b + 1 = 82$, $b^a + 1 = 65$

$a^b = 81, b^a = 64 \Rightarrow a = 3, b = 4$

$\therefore a^b = 81$ is largest

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set $\{10a+b:1\le a\le 5, 1\le b\le 5\}$. Where a,b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Sol. Ans: 47.

$10a+b,\ 1\le a\le 5,\ 1\le b\le 5$

Let a_n be the set of elements with $a=n$

$a_1=\{11,12,13,14,15\}$

$a_2=\{21,22,23,24,25\}$

$a_3=\{31,32,33,34,35\}$

$a_4=\{41,42,43,44,45\}$

$a_5=\{51,52,53,54,55\}$

According to the given condition, we can select any number from each of set a_1, a_2, a_3, a_4, a_5.

$\therefore N={}^5C_1\times{}^4C_1\times{}^3C_1\times{}^2C_1\times{}^1C_1$

$=5\times4\times3\times2\times1=120$

$N=120=73\times1+47$

$\therefore$ Remainder when N is divided by $73=47$

13. Consider the sequence

$1,7,8,49,50,56,57,343,\ldots,$

which consists of sums of distinct powers of 7, that is $7^0, 7^1, 7^0+7^1, 7^2,\ldots,$ in increasing order. At what position will 16856 occur in this sequence?

Sol. Ans: 36.

Given sequence is $1,7,8,49,50,56,57,143,\ldots$

Each term is expressed as the sums of $7^0, 7^1, 7^2,\ldots$

So, converting each term in the given sequence to base -7, we get,

$1,10,11,100,101,110,111,1000,\ldots.$

We noticed that they are consecutive list of binary numbers.

Where n^{th} term is equal to n in binary we have, $16856_{10}=(100100)_7$

If we consider 100100 to be binary and converting it to decimal we get 36

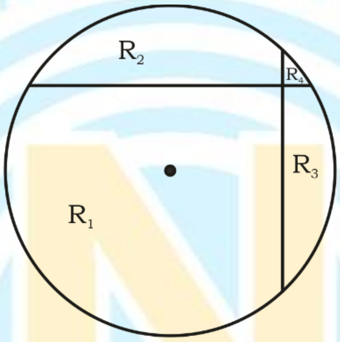
$\therefore 16856$ is in 36^{th} position

14. Let R denote the circular region in the $xy-$ plane bounded by the circle $x^2+y^2=36$. The lines $x=4$ and $y=3$ divide R into four regions $R_i, i=1,2,3,4$. If $[R_i]$ denotes the area of the region R_i and if $[R_1]>[R_2]>[R_3]>[R_4]$, determine $[R_1]-[R_2]-[R_3]+[R_4]$. (Here $[\Omega]$ denote the area of the region Ω in the plane.)

Sol. Ans: 48.

We have, $[R_1]+[R_2]+[R_3]+[R_4]=36\pi$

$\therefore [R_1]-[R_2]-[R_3]+[R_4]$



$=2([R_1]+[R_4])-([R_1]+[R_2]+[R_3]+[R_4]) = 2([R_1]+[R_4])-36\pi$

In $\Delta DAE,\ AE=2\sqrt{5}$

In $\Delta DFO,\ DF=3\sqrt{3}$

In $\Delta OFC,\ FC=3\sqrt{3} \Rightarrow CG=3\sqrt{3}-4$

In $\Delta OBE,\ BE=2\sqrt{5} \Rightarrow BG=2\sqrt{5}-3$

In $\Delta ODF, \angle ODF=\frac{\pi}{6}\left(\because \frac{OE}{OD}=\frac{3}{6}=\frac{1}{2}\right) \qquad \Rightarrow \angle DOF=\frac{\pi}{3}$

Let $\angle AOE=\alpha$

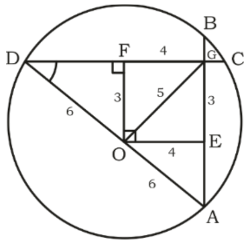
$\angle DOA=2\pi-\frac{\pi}{3}-\frac{\pi}{2}-\alpha=\frac{7\pi}{6}-\alpha$

$[R_1]=\left[\overline{DOA}\right]+[AOG]+[DOG]$

$=\frac{1}{2}\times 6^2\times\left(\frac{7\pi}{6}-\alpha\right)+\frac{1}{2}\times\left(3+2\sqrt{5}\right)\times 4+\frac{1}{2}\times\left(4+3\sqrt{3}\right)\times 3$

$= 21\pi - 18\alpha + 112 + 4\sqrt{5} + \dfrac{9\sqrt{3}}{2}$

In ΔOCH, $\angle COH = \dfrac{\pi}{6}\left(\because \dfrac{CH}{OC} = \dfrac{3}{6} = \dfrac{1}{2}\right)$



In ΔOBI, $\angle BOI = \dfrac{\pi}{2} - \alpha \left(\because \dfrac{4}{6} = \cos\alpha \text{ in } \Delta AOE\right)$

$\angle BOC = \dfrac{\pi}{2} - \dfrac{\pi}{6} - \left(\dfrac{\pi}{2} - \alpha\right) = \alpha - \dfrac{\pi}{6}$

$\therefore [R_4] = [\overline{OBC}] - [OBG] - [OGC]$

$= \dfrac{1}{2} \times 6^2 \times \left(\alpha - \dfrac{\pi}{6}\right) - \dfrac{1}{2} \times BG \times DE - \dfrac{1}{2} \times CG \times DF$

$= \dfrac{1}{2} \times 36 \times \left(\alpha - \dfrac{\pi}{6}\right) - \dfrac{1}{2} \times (2\sqrt{5} - 3) \times 4 - \dfrac{1}{2} \times (3\sqrt{3} - 4) \times 3$

$= 18\alpha - 3\pi - 4\sqrt{5} + 6 - \dfrac{9\sqrt{3}}{2} + 6 = 18\alpha - 3\pi + 12 - 4\sqrt{5} - \dfrac{9\sqrt{3}}{2}$

We have, $[R_1] - [R_2] - [R_3] + [R_4]$

$= 2([R_1] + [R_4]) - 36\pi = 2(24 + 18\pi) - 36\pi \quad = 48 + 36\pi - 36\pi = 48$

15. In base-2 notation, digits are 0 and 1 only and the places go up in powers of –2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base –2 how many non zero digits does it contain?

Sol. Ans: 6.

We have, $2019 = 2048 - 32 + 4 - 2 + 1$

$(4096 - 2048) - 32 + 4 - 2 + 1$

$(-2)^{12} + (-2)^{11} + (-2)^5 + (-2)^2 + (-2)^1 + (-2)^0 = (1100000100111)_{(2)}$

$\therefore$ number of *non*-zero digits $= 6$

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

Sol. Ans: 69.

Given, $p < 20 \Rightarrow p^2 < 400$

Also $\sqrt{n} < p \Rightarrow n < p^2 < 400 \Rightarrow n \in \{1, 2, ..., 399\}$

p	n	possible 'n'
2	<4	2
3	<9	3, 6
5	<25	5, 10, 15, 20
7	<49	7, 14, 21, 28, 35, 42
11	<21	11, 22, 33, 44, 55, 66, 77, 88, 99, 110
13	<169	13, 26, 39, ...,156
17	<289	17, 34, 57,...,272
19	<361	19, 38,...,342

No .of possible $N = 1+2+4+6+10+12+16+18 = 69$

17. Let a, b, c be distinct positive integers such that $b+c-a$, $c+a-b$ and $a+b-c$ are all perfect squares. What is the largest possible value of $a+b+c$ smaller than 100?

Sol. Ans: 91.

Let $b+c-a = p^2$ __(1)

$c+a-b = q^2$ ______(2)

$a+b-c = r^2$ _____(3)

$\because a, b, c$ are +ve integers

$\Rightarrow p, q, r$ are +ve integers

$(1)+(2)+(3) \Rightarrow a+b+c = p^2+q^2+r^2$

We have to find out the largest value of $a+b+c$ or $p^2+q^2+r^2$ less than 100

We have either p, q, r are all even or all odd.

Case i: If p, q, r are all even

Let us maximise $p^2+q^2+r^2$

So, let $b+c-a=8^2=64$

$c+a-b=4^2=16$

$a+b-c=2^2=4$

On solving we get $a=10, b=34, c=40$

$\therefore a+b+c=84$

Case: ii If p, q, r are all odd

So, let $b+c-a=9^2=81$

$c+a-b=3^2=9$

$a+b-c=1^2=1$

On solving we get $a=5, b=41, c=45$

$\therefore a+b+c=91$

Hence largest possible value of $a+b+c$ is 91

18. What is the smallest prime number p such that p^3+4p^2+4p has exactly 30 positive divisors?

Sol. Ans: 43.

$p^3+4p^2+4p=p\left(p^2+4p+4\right)=p(p+2)^2$

Given it has exaxtly 30 divisors. So, this is of the form:

a^1b^{14} or $a^2.b^9$ or $a^1.b^2.c^4$ or a^{29} for 30 divisions, we need 15 divisiors from $(p+2)^2$.

i.e. of the form $x^2.y^4$. So, we have to choose the number of the form $a^1.b^2.c^4$ we notices that for all $p<43$ has 5 divisors only

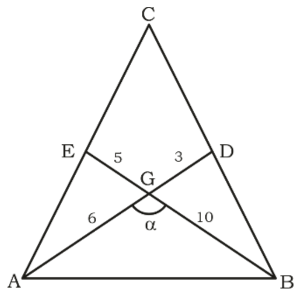
So, taking $p=43$, we get $(p+2)^2=45^2=(5\times 9)^2=5^2\times 3^4$

$\therefore p=43$ is the smallest

19. If 15 and 9 are lengths of two medians of a triangle. What is the maximum possible area of the triangle to the nearest integer?

Sol. Ans: 90.

Let $AD = 9 \Rightarrow AG = 6$, $BE = 15 \Rightarrow BG = 10$



We have $[AGB] = \frac{1}{2} \times AG \times BG \times \sin\alpha$

$= \frac{1}{2} \times 6 \times 10 \times \sin\alpha = 30\sin\alpha$

$\therefore [ABC] = 3 \times [AGB] = 3 \times 30\sin\alpha$

≤ 90

$\therefore \max.area = 90$

20. How many 4-digit numbers $\overline{abcd}$ are there such that $a < b < c < d$ and $b - a < c - b < d - c$?

Sol. Ans: 07.

We have $a, b, c, d \in \{0, 1, 2, ..., 9\}$

$\because a < b < c < d$

$\Rightarrow a \geq 1,\ b \geq 2,\ c \geq 3,\ d \geq 4$

Given, $b - a < c - b < d - c$

$b - a < c - b$ and $c - b < d - c$

$\Rightarrow 2b < a + c \Rightarrow 2c < b + d$

Case i: If $a = 1 \Rightarrow b = 2$ only

$c = 4$ *or* 5 $d = 7,\ 8$ *or* 9 [possibilities]

$\therefore$ possible numbers are:

$\overline{abcd} = 1247,\ 1248,\ 1249,\ 1259$

Case ii: If $a = 2 \Rightarrow b = 3$ only, $c = 5$ only, $d = 8$ *or* 9 [possibilities]

$\therefore$ possible numbers are

$\overline{abcd} = 2358, 2359$

Case iii: If $a = 3$

$\Rightarrow b = 4$, $c = 6$ only $d = 9$ only [possibilities]

$\therefore$ possible numbers are

$\overline{abcd} = 3469$

Total 7 numbers are possible

21. Consider the set E of all positive integers n such that when divided by 9, 10,11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

Sol. Block

22. In parallelogram ABCD, $AC = 10$ and $BD = 28$. The points K and L in the plane of $ABCD$ move in such a way that $AK = BD$ and $BL = AC$. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Sol. Ans: 02.

Produce CD to K' such that $CD = DK'$

Then $BDK'A$ is a parallelogram

$\because AB = CD = DK'$

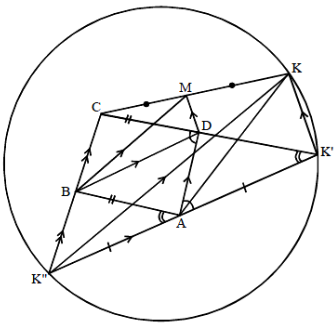
$AB || DK'$

$\therefore AK' = BD$

Draw a circle with centre A and radius BD which cuts CD produced at K' and CB produced at K'' then K'' AK' are collinear as $\angle CDA + \angle BAD = 180^0$

$\angle CDA = \angle DAK' + \angle DK'A = \angle DAK' + \angle BAK''$ $[\because BA \parallel DK']$

$\therefore \angle DAK' + \angle BAK'' + \angle BAD = 180^0$



Thus $K'AK''$ is a diameter

Lat K is nay point on this circle

Since M is a mid point of CK

D is a mid point of CK' then $MD \parallel KK'$

In $\Delta CK'K''$.

D is a mid-point of CK'

$DB \parallel K'A$ i.e. $DB \parallel K'K''$

Therefore, B is a mid-point of CK''

In $\Delta CK''K$

B, M are the mid points of CK'' and CK respectively.

Therefore, In ΔBMD and $\Delta K''KK'$

$BM \parallel K''K$

$MD \parallel KK'$

$BD \parallel K''K'$

$\therefore \angle BMD = \angle K''KK' = 90^0$

$\because K''K'$ is a diameter similarly for other triangle

$\angle ANC = 90^0$.

So, $\frac{\angle BMD}{2} = 45^0, \frac{\angle ANC}{2} = 45^0$

$\cot^2\left(\frac{\angle BMD}{2}\right) + \tan^2\left(\frac{\angle ANC}{2}\right) = 1 + 1 = 2.$

23. Let t be the area of regular pentagon with each side equal to 1. Let $P(x) = 0$ be the polynomial equation with least degree, having integer coefficients, satisfied by $x = t$ and the gcd of all the coefficient equal to1. If M is the sum of the absolute values of the coefficients of $P(x)$. What is the integer closest to $\sqrt{M}$? $\left(\sin 18^0 = \left(\sqrt{5} - 1\right)/2\right)$.

23. Block

24. For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$. Define $f(n) = \max\{m \in N \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + \ldots + f(10)$.

24. Ans 75.

Given that $a_1 = 9744$

$a_2 = 99744$ is divided by 16.

We have $a_1 = 9744 = 10^4 - 256 \equiv 0\left(\bmod 2^4\right)$

$a_2 = 99744 = 10^5 - 256 \equiv 0\left(\bmod 2^5\right)$

$a_3 = 999744 = 10^6 - 256 \equiv 0\left(\bmod 2^6\right)$

$a_4 = 10^7 - 256 \equiv 0\left(\bmod 2^7\right)$

$a_5 = 10^8 - 256 \equiv 0\left(\bmod 2^{13}\right)$

$a_6 = 10^9 - 256 \equiv 0\left(\bmod 2^8\right)$

$a_7 = 10^{10} - 256 \equiv 0\left(\bmod 2^8\right)$

$a_8 = 10^{11} - 256 \equiv 0\left(\bmod 2^8\right)$

$a_9 = 10^{12} - 256 \equiv 0\left(\bmod 2^8\right)$

$a_{10} = 10^{13} - 256 \equiv 0\left(\bmod 2^8\right)$

$f(1) + f(2) + \ldots . f(10)$

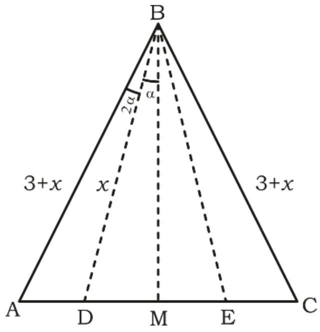
$4 + 5 + 6 + 7 + 13 + 8 + 8 + 8 + 8 + 8 = 75$

25. Let ABC be an isosceles triangle with $AB = BC$. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and $AB - BD = 3$. Find AC.

Sol. Let BD and BE are trisectors of $\underline{|ABC}$

Let $\underline{|ABC} = 6\alpha \Rightarrow \underline{|ABD} = 2\alpha$, $\Rightarrow \underline{|DBM} = \alpha$

$AB = BC \Rightarrow BM \perp AC$



Let $BD = x \Rightarrow AB = BC = x+3$

In $\Delta ABM,\ BM = (x+3)\cos 3\alpha = x.\cos\alpha$

$$= \frac{(x+3)(4\cos^3\alpha - 3\cos\alpha)}{\cos\alpha} = x$$

$$= (x+3)(4\cos^2\alpha - 3) = x. \Rightarrow \sin^2\alpha = \frac{3}{4(x+3)}$$

Now $AM = (3+x)\sin 3\alpha \Rightarrow AC = 2(3+x)(3\sin\theta - 4\sin^2\theta)$

$$= 2(3+x)\sin\theta\left(3 - 4\frac{3}{4(x+3)}\right) = 6(2+x)\frac{\sqrt{3}}{2\sqrt{x+3}} = 3(x+2)\frac{\sqrt{3}}{\sqrt{x+3}}$$

$$\Rightarrow x = 3y \Rightarrow AC = \frac{3(3y+2)}{\sqrt{y+1}}$$

$$\Rightarrow y+1 = z^2 \Rightarrow AC = \frac{3(3z^2-1)}{z} = 9x - \frac{3}{z} \Rightarrow z = 1\ \ or\ 3$$

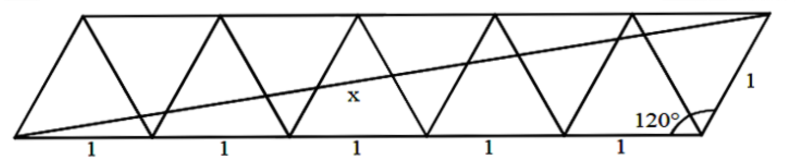
But $z \neq 1$ as $x = 0$ not possible

$\Rightarrow z = 3 \Rightarrow AC = 26$

26. A friction-less board has the shape of an equilateral of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N?

Sol. Ans:31

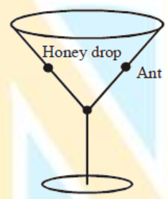
We have by cosine rule



$x^2 = 5^2 + 1^2 - 2.5.1\cos 120^0 = 31$

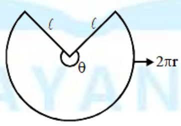
$x = \sqrt{31} \Rightarrow N = 31$

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass (see the figure). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass.)

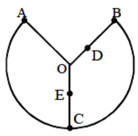


Sol. Ans: 38

Let cut the cone along slant height of cone.



Now, $\ell\theta = 2\pi r \quad \theta = \frac{4\pi}{3}$



When A goes B (fixed) the major arc becomes base circle of the cone i.e. may AB becomes full circle. So, mid-point of major AB i.e point C will be diametrically opposite end of fixed point B.

Now, Ant lies at the mid-point of OB(above the plane) i.e. point D and the she should reach point E (below the plane). So, required shortest path is D to C then C to E.

(Note $\angle DOC = \frac{2\pi}{3}$)

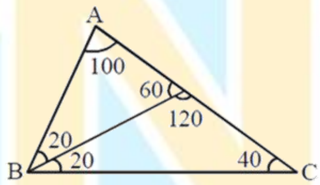
So $d = DC + CE \quad = \sqrt{l^2 + \frac{l^2}{4} + \frac{l^2}{2}} + \frac{l}{2} = \frac{\sqrt{7}l}{2} + \frac{l}{2} = \frac{(\sqrt{7}+1)21}{2}$

$[d] = 38.$

28. In a triangle ABC, it is known that $\angle A = 100^0$ and $AB = AC$. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10^0 \approx 0.174$.

Sol. Ans 27

In ΔBCD, by sine rule $\frac{BC}{\sin 120^0} = \frac{BD}{\sin 40^0} = \frac{20}{\sin 40^0}$



$BC = \frac{10\sqrt{3}}{\sin 40^0}.$

Given $\cos 80^0 = \sin 10^0 \approx 0.174$

$\sin 40^0 = \sqrt{\frac{1-\cos 80}{2}} \approx 0.643$

$BC = 27$

29. Let ABC be an acute angled triangle with $AB = 15$ and $BC = 8$. Let D be appoint on AB such that BD=BC. Consider points E on AC such that $\angle DEB = \angle BEC$. IF α denotes the product of all possible values of AE, find $[\alpha]$ the integer part of α.

Sol. Block

30. For any real number x, let $[x]$ denote the integer part of x; $\{x\}$ be the fractional part of x. $(\{x\}=x-[x])$. Let A denote the set of all real numbers x satisfying

$$\{x\}=\frac{x+[x]+\left[x+\left(\frac{1}{2}\right)\right]}{20}.$$

If S is the sum of all numbers in A, find $[S]$.

Sol. Ans 21

$$\{x\}=\frac{x+[x]+\left[x+\frac{1}{2}\right]}{20} \Rightarrow 20f=2I+f+\left[x+\frac{1}{2}\right]$$

Let $x=I+f=[x]+\{x\}$

Case:1 $0\le f<\frac{1}{2}\Rightarrow\left[x+\frac{1}{2}\right]=I$

So, $19f=3I\in[0,\frac{19}{2})\Rightarrow I\in[0,\frac{19}{6})$

$$x=I+f=I+\frac{3I}{19}=\frac{22I}{19}$$

$I=0,1,2,3.$

Case-2:

$$f\in[\frac{1}{2},1)\Rightarrow\left[x+\frac{1}{2}\right]=I+1$$

So, $19f=3I+1\in[\frac{19}{2},19)$

$\Rightarrow I\in[\frac{17}{6},6)$

$I=3,4,5$

$$x=I+f=I+\frac{3I+1}{19}=\frac{22I}{19}+\frac{1}{19}$$

Hence $S=\frac{22}{19}\times18+\frac{3}{19}=21$

$[S]=21$