PRMO_2019 **Date: 11-08-19**

Max.Marks:102 **FINAL-EXAM** **Max.Time:3 hrs**

KEY & SOLUTIONS

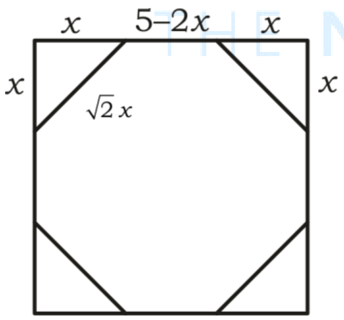
PRMO-KEY									
1	2	3	4	5	6	7	8	9	10
04	13	13	36	10	29	51	49	14	55
11	12	13	14	15	16	17	18	19	20
06	18	10	53	45	40	30	20	13	*
21	22	23	24	25	26	27	28	29	30
17	78	55	37	48	50	84	15	47	64

SOLUTIONS

01. We have,

$$\sqrt{2}x = 5 - 2x$$

$$\left(2+\sqrt{2}\right)x = 5$$



$$\therefore x = \frac{5}{2+\sqrt{2}}$$

Hence, area removed $= 4\times\left(\frac{1}{2}\times x\times x\right)$

$= \frac{4}{2} \times x^2$

$= 2x^2$

$= 2\left(\frac{5}{2+\sqrt{2}}\right)^2$

$\simeq 4.289$

$\therefore$ Nearest integer $= 4$

02. Given $f(x) = x^2 + ax + b$

$f\left(x+\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 + a\left(x+\frac{1}{x}\right) + b = \left(x^2+ax+b\right) + \left(\frac{1}{x^2}+\frac{a}{x}+b\right)$

$\Rightarrow x^2 + \frac{1}{x^2} + 2 + ax + \frac{a}{x} + b = x^2 + ax + b + \frac{1}{x^2} + \frac{a}{x} + b$

$\Rightarrow b = 2$

$\therefore f(x) = x^2 + ax + 2$

We have Δ should be a perfect square

$\therefore \Delta = a^2 - 8 = m^2$ for some $m \in \mathbb{Z}$

$a^2 - m^2 = 8$

$(a+m)(a-m) = 8$

8×1

4×2

If $a+m=8$ & $a-m=1$, then $a = \frac{9}{2}$ not an integer

If $a+m=4$ & $a-m=2$, then $a=3$ is an integer.

So, $a^2 + b^2 = 3^2 + 2^2 = 13$

03. Given $x_{n+1} = 1 + x_1x_2....x_{n-1}x_n$

Put $n=4$, $x_5 = 1 + x_1x_2x_3x_4$

$\Rightarrow x_1x_2x_3x_4 = 43 - 1 = 42$

$\therefore x_6 = 1 + x_1x_2x_3x_4x_5$

$= 1 + (x_1x_2x_3x_4)x_5$

$= 1 + 42 \times 43$

$= 1807$

$= 13 \times 139$

Largest prime factor $=139$

Sum of digits $=1+3+9=13$

04. After each distance of 4 feet, the ant turns 160^0

Let it travel n such distances

After n such distances total angle $160\times n$

This would be some integral multiple of 360^0

$\therefore 160^0\times n=360^0\times m$

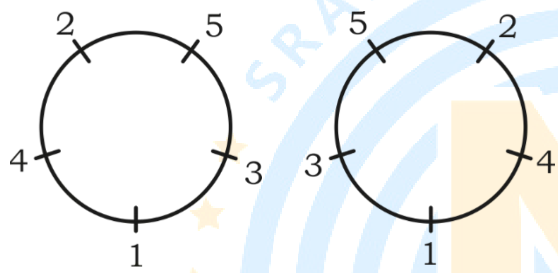
$4n=9m$

So, we have $n=9$

So, the ant travels 9 such distances of 4 feet

$\therefore$ Total distance $=9\times 4=36$ feet.

05. First fix the position of 2. The only possible seatings can be 24135 or 25314. In clock wise direction



There are 5 possible positions for 2.

Hence the required answer is $5\times 2=10$

06. $\because \overline{abc}$ is a 3-digit numbers $a,b,c<10$ & $a\neq 0$

Given, $a^2+b^2=c^2$

$\Rightarrow \overline{abc}=345$ or 435 $\left(\because 3^2+4^2=5^2\right)$

$345=3\times 5\times 23$

$435=3\times 5\times 29$

So, $\overline{abc}=435=3\times 5\times 29$ is the required 3- digit number

$\therefore$ Largest prime factor $=29$

07. Minute hand covers 6^0 in one minute and hour hand covers $\frac{1}{2}^0$ in one minute.

Let us consider, after x min, the minute hand and hour hand make 90^0-in clockwise direction.

Also let us consider, after y min, the minute hand and hour hand make 90^0 in anti-clockwise direction. (i.e in clock wise direction 270^0)

$6x-\frac{x}{2}=90^0$ & $6y-\frac{y}{2}=270^0$

$\Rightarrow x=\frac{180}{11}$ & $y=\frac{540}{11}$

$\therefore$ Difference $=y-x$

$=\frac{540}{11}-\frac{180}{11}$

$=\frac{360}{11}$

$=32\frac{8}{11}$

$\therefore a=32,\ b=8,\ c=11$

$a+b+c=32+8+11=51$

08. We have, ω & ω^2 are roots of $x^2+x+1=0$

Given, $x^{2^n}+x+1$ is divisible by x^2+x+1

$\Rightarrow 2^n$ should be of the form $3m+2$ $\left(\because \omega^2=1 \text{ \& } \omega,\ \omega^2 \text{ are roots}\right)$

$\Rightarrow 2^n$ is odd

So, $n=3,5,7,...99$

$\therefore 49$ numbers are possible

09. We have $\left|\frac{22}{7}-\frac{p}{q}\right|$ should be smallest for $q<100$

i.e. $\left|\frac{22q-7p}{7q}\right|$ should be smallest

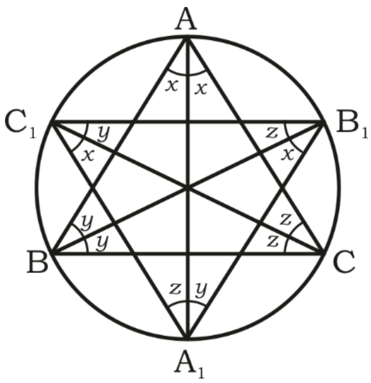
i.e. $|22q-7p|=1$

If $q=99,\ p=311$

$\therefore p-3q=311-3\times 99$

$=311-297=14$

10. $2x+2y+2z=180$



$x+y+z=90^0$

$\because A, B, C, A_1, B_1, C_1$ are concyclic

We have, $\angle A_1 = y+z$

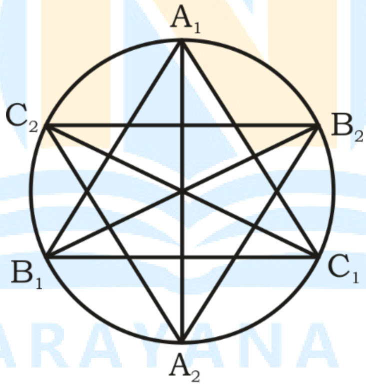
$\angle B_1 = x+z$

$\angle C_1 = x+y$

i.e $\angle A_1 = \frac{1}{2}(\angle B + \angle C)$

Similarly,

$\angle A_2 = \frac{1}{2}(\angle B_1 + \angle C_1)$



$= \frac{1}{2}(x+z+x+y)$

$= x + \frac{1}{2}(y+z)$

$= \frac{x}{2} + \frac{x+y+z}{2}$

$= \frac{40}{2} + \frac{90}{2}$

$= 10 + 45$

$= 55^0$

11. $\cos A.\cos B+\sin A.\sin B.\sin kC=1$

Case: I: If $kC\in\left(180^0, 360^0\right)$, then $\cos A.\cos B=1-\sin A.\sin B.\sin kC$

$=\cos A.\cos B\geq 1+\sin A.\sin B$

$(\because \sin kC$ is negative$)$

Impossible

Case:2: If $kC\in\left(0, 180^0\right)$

Then $\cos A.\cos B=1-\sin A.\sin B.\sin kC$

$\Rightarrow \cos A.\cos B+\sin A.\sin B\geq 1$ $(\because \sin kC$ is +ve$)$

$\Rightarrow \cos(A-B)\geq 1$

$\Rightarrow A=B$ & $kC=90^0$

$\because A=B\Rightarrow A+B+C=180^0$

$\Rightarrow 2A+C=180^0$

$\Rightarrow C$ should be an even integer

$90=2\times 3^2\times 5$

The even factors are $2, 2\times 3, 2\times 5, 2\times 3^2, 2\times 3\times 5$ and $2\times 3^2\times 5$

'6' possible values are there for C

12. Let $a_1=m_1^2, a_2=m_2^2, \ldots, a_k=m_k^2$

$\frac{1}{m_1}+\frac{1}{m_2}+\ldots.+\frac{1}{m_k}=1$

For $k=2$, $\frac{1}{m_1}+\frac{1}{m_2}\neq 1$

$\because 6=1+2+3]$

$\Rightarrow \frac{1}{6}+\frac{2}{6}+\frac{3}{6}=1$

$\Rightarrow \frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$

For $k=3$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$, so $k=3$ is good

Similarly, for $k=4, \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}=1$ $(\because 12=1+2+3+6)$

For $k=5, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}=1$ $(\because 24=1+2+3+6+12)$

We noticed that, if a number can be written as sum of its k distinct divisors, then k is good.

So, k is goof for all $k \ge 3$

$\therefore f(n) = 3+4+5+...+(n+2) = \frac{n(n+5)}{2}$

$\therefore \frac{f(n+5)}{f(n)} = \frac{(n+5)(n+10)}{n(n+5)}$

$= \frac{n+10}{n}$

$= 1 + \frac{10}{n}$

$\Rightarrow n = 1, 2, 5, 10$

$\therefore$ Required number $= 1+2+5+10 = 18$

13. We have, $(x_1 + x_2 + ... x_{101})^2 = x_1^2 + x_2^2 + ... + x_{101}^2 + 2 \sum_{1 \le i < j \le 101} x_i x_j$

$\Rightarrow \sum_{1 \le i < j \le 101} x_i x_j = \frac{(x_1 + x_2 + ... + x_{101})^2 - (x_1^2 + x_2^2 + x_{101}^2)}{2}$

$\because x_1^2 = x_2^2 = ... = x_{101}^2 = 1$

$\therefore x_1^2 + x_2^2 + ... + x_{101}^2 = 101$

$\Rightarrow \sum_{1 \le i \le j \le 101} x_i x_j = \frac{1}{2}\left[(x_1 + x_2 + ... + x_{101})^2 - 101\right]$

We have, $(x_1 + x_2 + ... + x_{101})^2$ is greater than 101 and a perfect square.

We have, $(x_1 + x_2 + ... + x_{101})^2 = 121$

$\Rightarrow \sum_{1 \le i < j \le 101} x_i x_j = \frac{1}{2}(121 - 101) = 10$

14. Let $9n + 7 = m^2$

$9n + 54 - 54 + 7 = m^2$

$9(n+6) - 47 = m^2$

$n + 6 = \frac{m^2 + 47}{9}$

We have, $m^2 + 47$ is a multiple of 9 and $\frac{m^2+47}{9}$ is a prime

$n + 6 = 5 + \frac{m^2 + 2}{9}$

By checking

$m^2 - 2 = 9, 18, 27, 36, 45, 54, 63, 72$

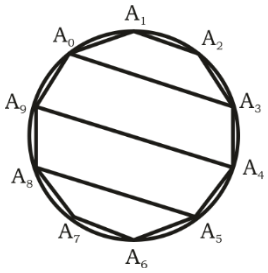
$\because n \geq 10$, we have $m = 22$

$\therefore n + 6 = 59$ is a prime and $9b + n = 484$ is a perfect square

$\therefore n = 53$

15. Case 1:

For vertex A_1 take A_2A_{10}, A_3A_9, A_4A_8 & A_5A_7 as, diagonals



So, there are ${}^4C_2 = 6$ ways to select a pair of parallel lines

Similarly, at all vertices total $6 \times 10 = 60$ pairs of parallel lines are possible

But we observed that at vertex A_6, again we are taking same pair of parallel lines.

So, total ways $= \frac{1}{2} \times 60 = 30$

Case:2: Consider the diagonals A_2A_9, A_3A_8, A_4A_1

Total ways $= \frac{{}^3C_2 \times 10}{2} = 15$

$\therefore$ Total no. of ways $= 30 + 15 = 45$

16. Let $Rs\ 'a'$ paying more

We have,

$13x + 17y = 10000$ .....(1)

$17x + 13y = 10000 + a$ .....(2)

$(1) + (2) \Rightarrow 30(x + y) = 20000 + a$ .....(3)

$(2) - (1) \Rightarrow 4(x - y) = a$ .....(4)

$x - y = \frac{a}{4}$

Let $a = 4k$

$30(x + y) = 20000 + 4k$

$x + y = \frac{20000 + 4k}{30}$

$= 666 + \frac{10+2k}{15}$

$\because |x-y|$ is min $\Rightarrow k = 10$

$\therefore a = 40$

17. $30a + 50b + 70c \le 343$

$3a + 5b + 7c \le 34.3$

$\Rightarrow 3a + 5b + 7c \le 34$

Let $a = 1 + p,\ b = 1 + q,\ c = 1 + r$

$\Rightarrow 3p + 5q + 7r \le 19$

$\because p, q, r$ are $+ve$ integers, n r take values 0,1 or 2 only

If $r = 0$, $3p + 5q \le 19$

If $q = 0 \Rightarrow p$ take 7 values

$q = 1 \Rightarrow p$ take 5 values

$q = 2 \Rightarrow p$ take 4 values

$q = 3 \Rightarrow p$ take 2 values

Total 18 values

If $r = 1$: $3p + 5q \le 12$

If $q = 0 \Rightarrow p$ take 5 values

$q = 1 \Rightarrow p$ take 3 values

$q = 2 \Rightarrow p$ take 1 values

Total 9 values

If $r = 2$,

$3p + 5q \le 5$

If $q = 0 \Rightarrow p$ take 2 values

$q = 1 \Rightarrow p$ take 1 values

Total 3 values

$\therefore$ In total $18 + 9 + 3 = 30$ solutions possible

18. Let $d = (a, b)$ $\quad a < b \Rightarrow p < q$

$\Rightarrow a = dp,\ b = dq$

Given,

$\frac{[a, b]}{(a, b)} = 495$

$\Rightarrow \frac{dpq}{d} = 495$

$\Rightarrow pq = 495 = 15\times 9\times 11$

p	q
5	55
9	55
11	45

Case:1: If $p = 5,\ q = 99$

Suppose $d = 20$ then $b = 1980 > 1000$

$\therefore$ No values are possible for a & b

Case:2: If $p = 9,\ q = 55$

$d = 12,13,14,15,16,17,18$ are possible

So, 7 pairs are possible

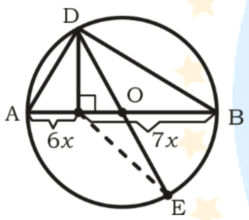
Case:3: If $p = 11,\ q = 45$

$d = 10,11,12,....,22$ are possible

So, 13 pairs are possible

Hence, in total 20 pairs of (a,b) are possible

19. Let $AC = 6x$



$BC = 7x$

Let r be the radius of the circle. Let $'O'$ be the centre of the circle.

We have,

$OC = 7x - r = r - 6x$

$\Rightarrow 2r = 13x$

$r = \dfrac{13x}{2}$

Join CE

$\because CO$ is median of ΔCDE

$\Rightarrow [DCO] = \dfrac{1}{2}[CDE]$

$\therefore [CDE] = CD\times CO$

Similarly $\because DO$ is median of ΔADB,

$[ABD] = 2[BDO]$

$= 2\times\frac{1}{2}\times r\times CD$

$= r\times CD$

$\therefore \frac{[ABD]}{[CDE]} = \frac{r\times CD}{CD\times CO} = \frac{r}{CO}$

$= \frac{r}{r-6x} = \frac{r}{r-6\times\frac{2r}{13}}$

$= 13$

20. Delete

21. Let the product of elements of A be m and that of B be n

We have,

$m\times n = 5\times6\times7\times8\times9$

$m+n$ is a prime

If $6\in A$, then $8,9\in A$ $(\because m+n \text{ is a prime})$

So, $A=\{6,8,9\}$, $B=\{5,7\}$

or $A=\{5,6,8,9\}$, $B=\{5\}$

or $A=\{5,6,7,8,9\}$, $B=\{7\}$

$\therefore m+n=(6\times8\times9)+(5\times7)$

or $(6\times7\times8\times9)+5$

or $(5\times6\times8\times9)+7$

$=467$ or 2029

or 2167

But 2029 & 2167 are not primes. Hence largest prime number possible is 467. Sum of digits of $N=4+6+7=17$

22. We have, $\frac{\sqrt{n}+\sqrt{n-1}}{2} < \sqrt{n} < \frac{\sqrt{n+1}+\sqrt{n}}{2}$

$\Rightarrow \frac{2}{\sqrt{n}+\sqrt{n-1}} > \frac{1}{\sqrt{n}} > \frac{2}{\sqrt{n+1}+\sqrt{n}}$

$\Rightarrow 2\left(\sqrt{n}-\sqrt{n-1}\right) > \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-\sqrt{n}\right)$

$\Rightarrow 2\left(\sqrt{n+1}-\sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{n-1}\right)$

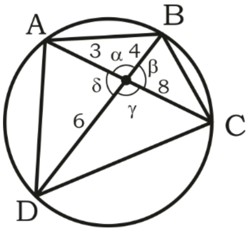
So, $2\left(\sqrt{1600}-1\right) < \sum_{n=1}^{1599}\frac{1}{\sqrt{n}} < 2\left(\sqrt{1599}-0\right)$

$\Rightarrow 78 < \sum_{n=1}^{1519} \frac{1}{\sqrt{n}} < 80$

Also $\Rightarrow \sum_{n=2}^{1600} \frac{1}{\sqrt{n}} < 2\left(\sqrt{1600}-1\right) < 78$

Hence $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}} = 78$

23. $\because ABCD$ is cyclic



$\Rightarrow PA \times PC = PB \times PD$

$\Rightarrow 3 \times 8 = 4 \times 6$

So, Let $PA = 3,\ PB = 4,\ PC = 8,\ PD = 6$

$[ABCD] = [ABP] + [BCP] + [CDP] + [DAP]$

$= \frac{1}{2} \times 3 \times 4 \times \sin\alpha + \frac{1}{2} \times 4 \times 8 \times \sin\beta$

Area is max. when $\alpha = \beta = \gamma = \delta = 90^0$

So, $\Delta_{\max} = \frac{1}{2}(12 + 32 + 48 + 18)$

$= 55$

24. We have, $f(n) = {}^nC_0.2^n + {}^nC_2.2^{n-2} + {}^nC_4.2^{n-4} + \ldots$

$\therefore f(9) = {}^9C_0.2^9 + {}^9C_2.2^7 + {}^9C_4.2^5 + {}^9C_6.2^3 + {}^9C_8.2^1$

$= \frac{(2+1)^9 + (2-1)^9}{2}$

$= \frac{3^9 + 1}{2}$

$f(3) = {}^3C_0.2^3 + {}^3C_2.2^1$

$= \frac{(2+1)^3 + (2-1)^3}{2} = \frac{3^3+1}{2}$

$\therefore \frac{f(9)}{f(3)} = \frac{3^9+1}{3^3+1}$

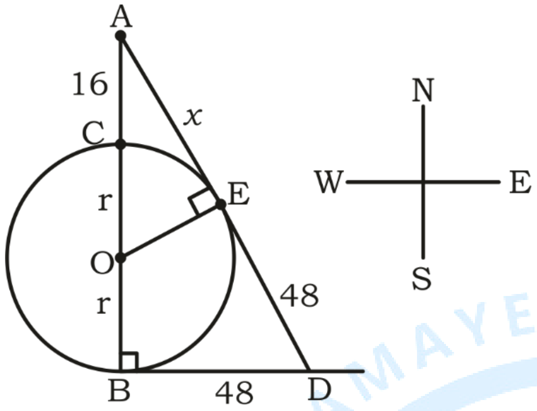
$= \frac{27^3+1}{27+1} = 27^2 - 27 + 1$

$= 703$

$= 19 \times 37$

$\therefore$ Largest prime factor $= 37$

25. Let r be the radius $BD = DE = 48$



Let $AE = x$

$\because \angle OAE = \angle BAD$ and $\angle AEO = 90^0 = \angle ABD$

$\Delta OAE \sim \Delta DAB$

$\Rightarrow \frac{OA}{AD} = \frac{AE}{AB} = \frac{OE}{BD}$

$\frac{16+r}{48+x} = \frac{x}{2r+16} = \frac{r}{48}$

$\frac{16+r}{48+x} = \frac{r}{48}$, $\frac{x}{2r+16} = \frac{r}{48}$

$\Rightarrow x = \frac{48(16+r)}{r} - 48$, $x = \frac{r(2r+16)}{48}$

$\therefore \frac{48(16+r) - 48r}{r} = \frac{r(2r+16)}{48}$

$r^2(8+r) = 24 \times 48 \times 16$

$= 24 \times 24 \times 32$

$= 24^2(24+8)$

$\therefore r = 24$

Hence diameter $= 48$

26. Given $xy + z = 160$

$x + yz$ is min. when y, z are as min as possible

by hit and trial,

$z = 1 \Rightarrow xy = 159$ not possible (prime)

$z = 2 \Rightarrow xy = 158 \Rightarrow x = 79, y = 2 \;\; \therefore x + yz = 83$

$z = 3 \Rightarrow xy = 157$ not possible

$z = 4 \Rightarrow xy = 156 \Rightarrow x = 26,\ y = 6$

$x + yz = 26 + 24 = 50$

For all other z, we will get $x + yz > 50$ therefore, minimum value of $x + yz = 50$

27. First put the $2A'S$

Total 6 ways

WLOG, let us assume the $A's$ are in the first and second place

$\underline{A}\ \underline{A}\ \_\ \_\ \_\ \_$

$B\ H\ A\ R\ A\ T$

Case: 1

If R is kept in $T's$ place, then T can go to 3 places (previous R and the $2A's$)

B, H occupies the other two places

Hence the no. of possibilities here are $1 \times 3 \times 2 = 6$

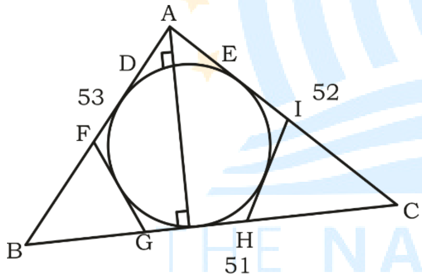
Case-2:

If R goes to the places occupied by A, then T has only 2 places and B, H again occupy other 2 places

Hence the no. of possibilities here are $2 \times 2 \times 2 = 8$

Thus, the answer is $6 \times (6 + 8) = 6 \times 14 = 84$

28. Let ABC be a parallel with $AB = 51$, $BC = 52$ and $AC = 53$



Let $DE \parallel BC$, $FG \parallel AC$ and $HI \parallel BA$

we have,

$\Delta ADE \sim \Delta FBG \sim \Delta IHC \sim \Delta ABC$

Let $DE = a$, $FG = b$ and $HI = c$

we have,

$\frac{r_1}{r} = \frac{a}{51} \Rightarrow r_1 = r.\frac{a}{51}$

Similarly,

$r_2 = r.\frac{b}{52}$ & $r_3 = r.\frac{c}{53}$

$\therefore r_1 + r_2 + r_3 = r\left[\frac{a}{51} + \frac{b}{52} + \frac{c}{53}\right]$

Let h_1 be the altitude from A

h_2 be the altitude from B

h_3 be the altitude from C

$\because \Delta ADE \sim \Delta ABC$

$\Rightarrow \frac{DE}{BC} = \frac{a}{51} = \frac{h_1 - 2r}{h_1} = 1 - \frac{2r}{h_1}$

Similarly,

$\frac{b}{52} = \frac{h_2 - 2r}{h_2} = 1 - \frac{2r}{h_2}$ and

$\frac{c}{53} = 1 - \frac{2r}{h_3}$

$\therefore r_1 + r_2 + r_3 = r\left[1 - \frac{2r}{h_1} + 1 - \frac{2r}{h_2} + 1 - \frac{2r}{h_3}\right]$

$= r\left[3 - 2r\left(\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}\right)\right]$

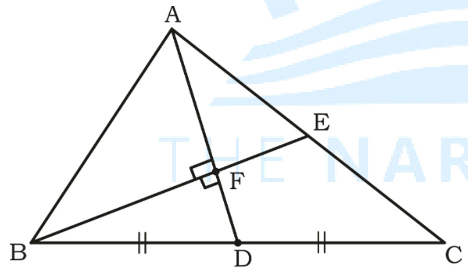
Largest integer that does not exceed $r_1 + r_2 + r_3$ is r

$r = \frac{\Delta}{s} = \frac{1170}{78} = 15$

29. Let $F = AD \cap BE$

$\Delta AFB \cong \Delta DFB$

$\left[\because BF = BF,\ \angle FBA = \angle FBD,\ \angle AFB = 90^0 = \angle DFB\right]$



$\Rightarrow AF = FD = \frac{7}{2}$ and

$AB = BD = \frac{1}{2}BC \Rightarrow \frac{AB}{BC} = \frac{1}{2}$

$\because BE$ is angular bisector

By $VABT$,

$\frac{CE}{AE} = \frac{BC}{AF} = \frac{2}{1}$

$\therefore [ABC] = 3[ABE]$

$= 3 \times \frac{1}{2} \times BE \times AF$

$= 3 \times \frac{1}{2} \times 9 \times \frac{7}{2}$

$= \frac{189}{4} = 47.25$

$\therefore$ Nearest integer $= 47$

30. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

It can be partitions into 3 sets of equal sum

$\Rightarrow$ either n or $(n+1)$ should be divisible by 3

i.e $n = 3m$ or $3m - 1$

If $n = 3m$, then $n = 6, 9, 12, \ldots, 99$

32 elements

If $n = 3m - 1$, then $n = 5, 8, 11, \ldots, 98$, 32 elements

$\therefore$ total terms in $E = 64$