# RMO-2019-Solutions

01. Suppose $x$ is a nonzero real number such that both $x^5$ and $20x+\frac{19}{x}$ are rational numbers. Prove that $x$ is a rational number.

Sol. Given that $x^5$ and $20x+\frac{19}{x}$ are rational.

Let $x^5 = a$

$$20x+\frac{19}{x}=b$$

$$\Rightarrow \quad 20x^2+19=bx$$

$$\Rightarrow \quad x^2=\frac{bx-19}{20}$$

So, $x^3=\frac{x(bx-19)}{20}=\frac{bx^2-19x}{20}$

$$=\frac{b\left(\frac{bx-19}{20}\right)-19x}{20}=\frac{b^2x-19b-380x}{400}$$

$$=\frac{(b^2-380)x}{400}-\frac{19b}{400}=cx+d$$

$(\because b$ is rational $\Rightarrow c,d$ are rational$)$

$$x^4=x^3\cdot x=(cx+d)\cdot x$$

$$=cx^2+dx=c\left(\frac{bx-19}{20}\right)+dx$$

$$=\frac{(bc+20d)x}{20}-\frac{19c}{20}=ex+f$$

$(\because b, c, d$ are rational $\Rightarrow e, f$ are rational$)$

$$x^5 = x^4 \cdot x$$

$$= (ex + f) \cdot x$$

$$= ex^2 + fx = e\left(\frac{bx - 19}{20}\right) + fx$$

$$= \frac{(be + 20f)x}{20} - \frac{19e}{20} = gx + f$$

$(\because b, e, f$ are rational $\Rightarrow g$ is rational$)$

Given $x^5 = a$ is rational

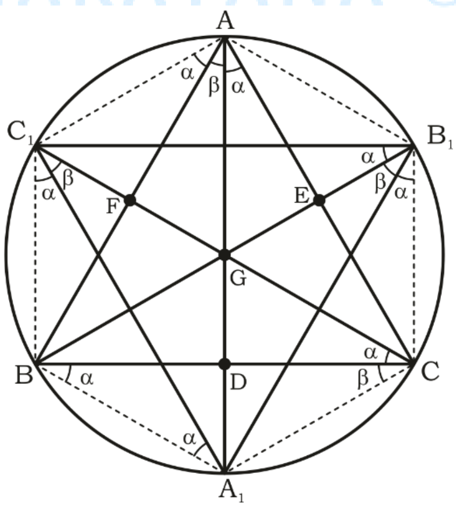
$\therefore x^5 = a = 9x + f$ is rational.

$\because g \,\&\, f$ are rational, $x$ should be rational

Hence Proved.

02. Let $ABC$ be a triangle with circumcircle $\Omega$ and let $G$ be the centroid of triangle $ABC$. Extend $AG$, $BG$ and $CG$ to meet the circle $\Omega$ again in $A_1$, $B_1$ and $C_1$, respectively. Suppose $\angle BAC = \angle A_1B_1C_1, \angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that $ABC$ and $A_1B_1C_1$ are equilateral triangles.

Sol.




Let $D, E, F$ are midpoints of $BC, AC$ and $AB$ respectively

Given, $\angle BAC = \angle A_1B_1C_1 \Rightarrow BC = A_1C_1$

$\angle ABC = \angle A_1C_1B_1 \Rightarrow AC = A_1B_1$

$\angle ACB = \angle B_1A_1C_1 \Rightarrow AB = B_1C_1$

Let $\angle A_1AC = \alpha$

$\Rightarrow \angle A_1B_1C = \angle A_1C_1C = \angle A_1BC = \alpha$

Let $\angle BAA_1 = \beta$

$\Rightarrow \angle BC_1A_1 = \angle BB_1A_1 = \angle BCA_1 = \beta$

$\because \angle BAC = \alpha + \beta = \angle A_1B_1C_1$

$\Rightarrow \angle BB_1C_1 = \alpha = \angle BAC_1 = \angle BCC_1 = \angle BA_1C_1$

We noticed that,

$\angle A_1BC = \alpha = \angle BCC_1$

$\Rightarrow A_1B \parallel CC_1$

Similarly, we can obtain

$B_1C \parallel AA_1$ and $AC_1 \parallel BB_1$

$\because BB_1 \parallel AC_1 \Rightarrow GE \parallel AC_1$

So, in $\Delta ACC_1$,

$\because GE \parallel AC_1$ and $E$ is midpoint of AC

by midpoint theorem,

$G$ is the midpoint of $CC_1$

Similarly, we can prove,

$G$ is the midpoint of $AA_1$ and $BB_1$.

i.e., $GA = GB = GC$

$\Rightarrow$ $G$ is the circumcentre of $\Delta ABC$

$\therefore \Delta ABC$ is equilateral.

Since $\Delta ABC \sim \Delta B_1C_1A_1$

$\Rightarrow$ $\Delta A_1B_1C_1$ is also equilateral,

Hence proved.

03. Let $a, b, c$ be positive real numbers such that $a+b+c=1$. Prove that:

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}$$

Sol. Consider:

$$a^2+b^3+c^3 = a^2\cdot 1+b^3+c^3$$

$$= a^2\cdot(a+b+c)+b^3+c^3$$

$$= a^3+b^3+c^3+a^2b+a^2c$$

$\because$ $a, b, c$ are positive real numbers

by A.M – G.M inequality,

$$\frac{a^3+b^3+c^3+a^2b+a^2c}{5} \ge \sqrt[5]{a^3\cdot b^3\cdot c^3\cdot a^2b\cdot a^2c}$$

$$\Rightarrow \frac{a^3+b^3+c^3+a^2b+a^2c}{5} \ge \left(a^7\cdot b^4\cdot c^4\right)^{\frac{1}{5}}$$

$$\Rightarrow \frac{1}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{1}{5\left(a^7\cdot b^4\cdot c^4\right)^{\frac{1}{5}}}$$

So, $$\frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{a}{5\left(a^7\cdot b^4\cdot c^4\right)^{\frac{1}{5}}}$$

Similarly we get,

$$\frac{b}{b^2+c^3+a^3} \le \frac{b}{5\left(a^4 \cdot b^7 \cdot c^4\right)^{\frac{1}{5}}}$$

and $\dfrac{c}{c^2+a^3+b^3} \le \dfrac{c}{5\left(a^4 \cdot b^4 \cdot c^7\right)^{\frac{1}{5}}}$

Also, $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc} \Rightarrow abc \le \dfrac{1}{27}$.

$\therefore$ LHS

$$\frac{a}{a^2+b^2+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+a^3+b^3}$$

$$\le \frac{a}{5\left(a^7 \cdot b^4 \cdot c^4\right)^{\frac{1}{5}}}+\frac{b}{5\left(a^4 \cdot b^7 \cdot c^4\right)^{\frac{1}{5}}}+\frac{c}{5\left(a^4 \cdot b^4 \cdot c^7\right)^{\frac{1}{5}}}$$

$$\le \frac{1}{5abc}\left[a^{3/5} \cdot b^{1/5} \cdot c^{1/5}+a^{1/5} \cdot b^{3/5} \cdot c^{1/5}+a^{1/5} \cdot b^{1/5} \cdot c^{3/5}\right]$$

$$\left[\begin{array}{l} \because \dfrac{a+a+a+b+c}{5} \ge \sqrt[5]{a^3 \cdot b \cdot c} \\ \Rightarrow \sqrt[5]{a^3 \cdot b \cdot c} \le \dfrac{3a+b+c}{5} \\ \Rightarrow a^{3/5} \cdot b^{1/5} \cdot c^{1/5} \le \dfrac{3a+b+c}{5} \\ \Rightarrow a^{1/5} \cdot b^{3/5} \cdot c^{1/5} \le \dfrac{a+3b+c}{5} \\ \Rightarrow a^{1/5} \cdot b^{1/5} \cdot c^{3/5} \le \dfrac{a+b+3c}{5} \end{array}\right]$$

So, LHS $\le \dfrac{1}{5abc}\left[\dfrac{3a+b+c}{5}+\dfrac{a+3b+c}{5}+\dfrac{a+b+3c}{5}\right]$

$$\le \frac{1}{25abc} 5(a+b+c) \le \frac{1}{5abc}.$$

04. Consider the following $3 \times 2$ array formed by using the numbers 1, 2, 3, 4, 5, 6:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable $k$, adding more columns, using the numbers 7, 8, 9, ..., $3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \text{ and } \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2 .$$

Sol. Let the numbers be $n, n+1, n+2, \ldots$ by arranging the numbers in a logical way such that the sum of the numbers in each row are equal

$$\begin{bmatrix} \bullet n & n+5 & n+7 \\ n+1 & \bullet n+3 & n+8 \\ n+2 & n+4 & \bullet n+6 \end{bmatrix} \qquad \ldots(1)$$

Sum of numbers in each row = $3n+12$

Sum of the square of row 1: $3n^2+24n+74$

row 2: $3n^2+24n+74$

row 3: $3n^2+24n+56$

we noticed that row 3 is short by 18.

Similarly we take the next 9 numbers from $n+9$ to $n+17$

In the similar way, we arrange the numbers. After arranging just interchange row 2 and row 3. So that row 2 and row 3 sums of the squares are equal

$$\begin{bmatrix} \bullet n+9 & n+14 & n+16 \\ n+10 & \bullet n+12 & n+17 \\ n+11 & n+13 & \bullet n+15 \end{bmatrix}$$

Interchange row 2 and row 3

$$\begin{bmatrix} n+9 & n+14 & n+16 \\ n+11 & n+13 & n+15 \\ n+10 & n+12 & n+17 \end{bmatrix} \quad \ldots(2)$$

Similarly take next 9 numbers, $n+18$ to $n+26$, arrange in the same manner.

After arranging, interchange row 1 and row 3.

i.e., $\begin{bmatrix} \bullet n+18 & n+23 & n+25 \\ n+19 & \bullet n+21 & n+26 \\ n+20 & n+22 & \bullet n+24 \end{bmatrix}$

Interchange row 1 and row 3

$$\begin{bmatrix} n+20 & n+22 & n+24 \\ n+19 & n+21 & n+26 \\ n+18 & n+23 & n+25 \end{bmatrix} \quad \ldots(3)$$

Finally, if we arrange (1), (2) and (3) we obtain 3 × 9 numbers of the required condition.

$$\begin{bmatrix} n & n+5 & n+7 & n+9 & n+14 & n+16 & n+20 & n+22 & n+24 \\ n+1 & n+3 & n+8 & n+11 & n+13 & n+15 & n+19 & n+21 & n+26 \\ n+2 & n+4 & n+6 & n+10 & n+12 & n+17 & n+18 & n+23 & n+25 \end{bmatrix}$$

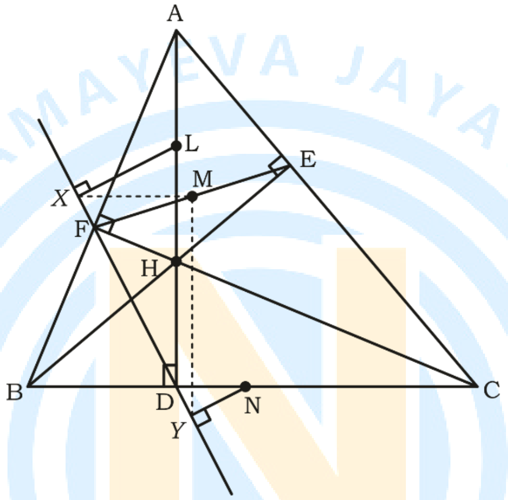
Put $n=1$, we get $3\times 9$ arrays as:

$$\begin{bmatrix} 1 & 6 & 8 & 10 & 15 & 17 & 21 & 23 & 25 \\ 2 & 4 & 9 & 12 & 14 & 16 & 20 & 22 & 27 \\ 3 & 5 & 7 & 11 & 13 & 18 & 19 & 24 & 26 \end{bmatrix}.$$

05. In an acute angled triangle $ABC$, let $H$ be the orthocentre, and let $D$, $E$, $F$ be the feet of altitudes from $A$, $B$, $C$ to the opposite sides, respectively. Let $L$, $M$, $N$ be midpoints of segments $AH$, $EF$, $BC$, respectively. Let $X$, $Y$ be feet of altitudes from $L$, $N$ on to the line $DF$. Prove that $XM$ is perpendicular to $MY$.

Sol. **Observation – 1:**

From nine point circle theorem, we have $L, D, E, F, N$ are concyclic with $LN$ as diameter.




**Observation – 2:**

$\because \angle AEB = 90^0 = \angle ADB$

$\Rightarrow AEDB$ is cyclic

Similarly, $BFEC$, $AFDC$ are cyclic

**Observation – 3:**

We have,

$\angle EFN = \angle EDC = 90 - \angle ADE$

$= 90 - \angle ABE$

$= 90 - (90 - \angle A)$

$= \angle A$

$$\angle FEN = \angle FDB = 90 - \angle ADF$$

$$= 90 - \angle ACF$$

$$= \angle A$$

$$\Rightarrow \angle EFN = \angle FEN = \angle A$$

$\therefore$ $\Delta EFN$ is isosceles

**Observation – 4:**

$$\angle NFL = \angle NEL$$

$$= 90^0$$ ($\because LN$ is a diameter)

In $\Delta LEF$,

$$\angle LFE = \angle NFL - \angle NFE$$

$$= \angle NEL - \angle NEF$$ ($\because \Delta EFN$ is isosceles)

$$= \angle LEF$$

$\therefore$ $\Delta LEF$ is isosceles

**Observation – 5:**

We have, $\angle NYF = \angle NMF = 90^0$ ($\because \Delta NEF$ is isosceles and $M$ is midpoint)

Also, $\angle LYF = \angle LMF = 90^0$ ($\because \Delta LEF$ is isosceles and $M$ is midpoint)

$\Rightarrow NYFM$ and $LXFM$ are cyclic

So, $\angle MXF = \angle MLF = 90 - \angle LFM = \angle A$

$$\angle MYF = \angle MNF = 90 - \angle NFM = 90 - \angle A$$

$$\therefore \angle MXF + \angle MYF = 90^0$$

$$\Rightarrow \angle XMY = 180 - (\angle MXF + \angle MYF) = 180 - 90 = 90^0$$

i.e., $XM \perp MY$; Hence proved.

06. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers $a$, $b$, $c$, $d$ among them such that $\gcd(a,b)=\gcd(b,c)=\gcd(c,d)=\gcd(d,a)=1$.

Sol. Construct a set of 91 points in plane $A_1, A_2, \ldots, A_{91}$

Two points are co-prime if they have an edge connecting them.

Consider the pair $A_i A_j$ to be good if there exists a path from $A_i$ to $A_j$ via one other point.

The number of good pairs can be maximum of ${}^{91}c_2$

Let $x_i$ be the number of edges at $A_i$

$\Rightarrow$ Number of good pairs corresponding to $A_i = {}^{x_i}c_2 \text{ (or) } \dfrac{x_i^2 - x_i}{2}$

By Cauchy-Schartz inequality,

$\therefore$ Total good pairs:

$$\sum \frac{x_i^2 - x_i}{2} \geq \frac{1}{2}\left(\frac{\sum x_i^2}{91} - \sum x_i\right)$$

$\because \sum x_i \geq 912$

$$\Rightarrow \sum \frac{x_i^2 - x_i}{2} \geq \frac{1}{2}\left(\frac{912^2}{91} - 912\right)$$

$$\geq \frac{1}{2} \times 912\left(\frac{912}{91} - 1\right)$$

$> 456 \times 9$

$> 455 \times 9 = {}^{91}c_2$

Hence, some of the 2 good pairs are connected twice.

Hence, there exists a good pair $A$, $B$ having pair through $C$, $D$.